## APPENDIX A

## FORMALISM OF NONLINEAR OPTICS

The polarization induced in a medium by an applied electric field $E(\bar{r},t)$ can be written in the form [1]

$$P_i(t) = \int_{-\infty}^{\infty} \chi_{ij}^{(1)}(t-\tau_1)E_j(\tau_1)d\tau_1$$

$$+ \int_{-\infty}^{\infty} \chi_{ijk}^{(2)}(t-\tau_1, t-\tau_2)E_j(\tau_1)E_k(\tau_2)d\tau_1 d\tau_2$$

$$+ \int_{-\infty}^{\infty} \chi_{ijkl}^{(3)}(t-\tau_1,t-\tau_2,t-\tau_3)E_j(\tau_1)E_k(\tau_2)E_l(\tau_3)d\tau_1 d\tau_2 d\tau_3 \ldots \quad (A.1)$$

This is simply the most general time-invariant constitutive relation between the vectors $P(t)$ and $E(t)$. Here $\chi^{(1)}$, $\chi^{(2)}$ and $\chi^{(3)}$ are the linear, second-order and third order susceptibilities respectively, and the indices i, j, k, l refer to the spatial coordinates x, y, z. In each term we sum over repeated indices. For media possessing inversion symmetry, terms in the above expansion involving even powers of E are ruled out. Since we shall be dealing primarily with nonlinear effects in isotropic media, we will focus on the third order term $P^{(3)}(r,t)$ which involves the fourth-rank tensor $\chi^{(3)}$.

Following Maker and Terhune [2] we now assume a set of discrete Fourier components for E(t) and P(t) so that these fields can be written

$$E(\bar{r},t) = \sum_{j=-N}^{N} \frac{E(\omega_j,r)}{2} e^{i(k_j r-\omega_j t)} \tag{A.2}$$

$$P(\bar{r},t) = \sum_{k=-M}^{M} \frac{P(\omega_k,r)}{2} e^{-i\omega_k t} \tag{A.3}$$

with $\omega_{-j} \equiv -\omega_j$, $k_{-j} \equiv -k_j$ and $E(\omega_{-j},r) \equiv E^*(\omega_j,r)$. Using (A.2) and (A.3) in (A.1), the Fourier component of the third order polarization at the frequency $\omega_1 = \omega_2 + \omega_3 + \omega_4$ is given by

$$P_i^{(3)}(\omega_1,r)$$
$$= D\chi_{ijkl}^{(3)}(-\omega_1,\omega_2,\omega_3,\omega_4)E_j(\omega_2)E_k(\omega_3)E_l(\omega_4)e^{i(k_2+k_3+k_4)r} . \tag{A.4}$$

D is a degeneracy factor whose value is 1, 3, or 6 depending on whether all, two, or none of the frequencies $\omega_2$, $\omega_3$ and $\omega_4$ are equal.

$\chi^{(3)}$ being a fourth-rank tensor, has in general $3^4 = 81$ components. However, symmetry relations reduce the number of independent components considerably. In particular, for isotropic media the 81-element tensor reduces to a set of three independent elements: $\chi^{(3)}_{1221}$, $\chi^{(3)}_{2121}$, $\chi^{(3)}_{1122}$ where the subscripts 1 and 2 refer to x, y or z [1]. All the other components can be expressed in terms of these three quantities. For example in the spatially degenerate case where i = j = k = 1, we find

$$\chi^{(3)}_{1111} = \chi^{(3)}_{1221} + \chi^{(3)}_{1212} + \chi^{(3)}_{1122} \quad . \tag{A.5}$$

In the more general case involving the interaction of waves with different polarizations, Maker and Terhune show that the nonlinear polarization can be written

$$\overline{P}(\omega) = A(\overline{E}\cdot\overline{E}^*)\overline{E} + \frac{1}{2}B(\overline{E}\cdot\overline{E})\overline{E}^* \tag{A.6}$$

where

$$A = 3[\chi^{(3)}_{1122}(-\omega,\omega,\omega,-\omega) + \chi^{(3)}_{1212}(-\omega,\omega,\omega,-\omega)]$$

and

$$B = 6\chi^{(3)}_{1221}(-\omega,\omega,\omega,-\omega).$$

In the next section we discuss some effects due to this third order polarization which are relevant to this dissertation.

## Phenomenology of $\chi^{(3)}$ (The Third-Order Nonlinear Susceptibility)

### The Slowly-Varying Envelope Approximation

The nonlinear polarization acts as a source term in Maxwell's equation and gives rise to a plethora of nonlinear effects. To describe these effects we start with the wave equation

$$\frac{\partial^2 E}{\partial Z^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(E + 4\pi P) \tag{A.7}$$

where

$$E = \text{Re } E(\omega,r)e^{-i\omega t} \quad \text{and} \quad P = \text{Re } P(\omega,r)e^{-i\ t}$$

and the complex envelope of the polarization density is given by

$$P_i = \omega_{ij}^{(1)}(\omega)E_j + P_i^{(3)}(\omega) \quad . \tag{A.8}$$

Defining the refractive index as $n^2 = 1 + 4\pi\omega$ and using Eq. (A.8) in (A.7) leads to

$$\frac{d^2E(\omega)}{dz^2} + \frac{\omega^2 n^2 E}{c^2} = \frac{-4\pi\omega^2}{c^2} P_i^{(3)}(\omega) \tag{A.9}$$

Since waves propagating in the z-direction are possible solutions of Eq. (A.9) we remove the rapidly-varying phase factor exp (ikz) from the solution by writing the fields as

$$E(\omega,z) = E(\omega,z)e^{ikz} \tag{A.10}$$

$$P(\omega,z) = P(\omega,z)e^{ikz} \tag{A.11}$$

where $k = \frac{n\omega}{c}$. The amplitudes E and P are now assumed to vary little within an optical cycle so that $\left|\frac{d^2E}{dz^2}\right| << \left|k\frac{dE}{dz}\right|$. Using this approximation (known as the slowly varying envelope approximation) we obtain the following first order differential equation for the field amplitudes:

$$2ik\frac{dE}{dz} = \frac{-4\pi k^2}{n^2} P^{(3)}(\omega) \quad . \tag{A.12}$$

This equation forms the basis for the analysis in the following sections of self-induced refractive index changes, optically induced birefringence, and four-wave mixing. The approach used is that of Maker and Terhune [2].

## Intensity-Dependent Refractive Index

The third order susceptibility $\chi^{(3)}$ leads to an intensity-dependent refractive index that results in self-focussing, self-phase modulation and optical bistability, among other effects. To calculate the size of the index change due to a given applied electric field we use the polarization

$$P = 3\chi^{(3)}_{1111}|E|^2E \tag{A.13}$$

in the slowly varying amplitude equation (Eq. A.12) and obtain

$$\frac{dE}{dz} = \frac{i2\pi k_{\omega}}{\varepsilon_{\omega}} \chi^{(3)}(-\omega,\omega,\omega,-\omega)|E|^2 E \tag{A.14}$$

Writing $E = |E|e^{i\psi}$, we find from (A.14) that

$$\frac{d|E|}{dz} = 0 \tag{A.15}$$

$$\frac{d\psi}{dz} = \frac{2\pi k}{\varepsilon_{\omega}} 3\chi^{(3)}|E|^2 \tag{A.16}$$

Equation (A.15) shows that in the absence of loss the field amplitude stays constant and from Eq. (A.16) we find that the index change due to the field is

$$\delta n = \frac{6\pi}{n} \chi^{(3)}_{1111}|E|^2 \tag{A.17}$$

## Optically Induced Birefringence

We have seen how the nonlinear polarization induced by a monochromatic wave can affect the propagation of the wave itself. More generally, a wave at one frequency $\omega$ can induce a birefringence for another wave at $\Omega$. The polarization responsible for this effect is given by

$$P_i(\omega) = 6\chi^{(3)}_{1221}(-\omega,\omega,\Omega,-\Omega)E_j(\omega)E_j(\Omega)E^*_i(\Omega)e^{ikz}$$

$$+ 6\chi^{(3)}_{1212}(-\omega,\omega,\Omega,-\Omega)E_j(\omega)E^*_j(\Omega)E_j(\Omega)e^{ikz}$$

$$+ 6\chi^{(3)}_{1122}(-\omega,\omega,\Omega,-\Omega)\ E(\Omega)\ ^2E_i(\omega)e^{ikz}\ . \qquad \text{(A.18)}$$

Once again, using this polarization in Eq. (A.12) we find that the changes in refractive index at $\omega$ parallel and perpendicular to the direction of polarization of $E(\Omega)$ are

$$\delta n_{\parallel} = \frac{12\pi}{n}\left\{\chi^{(3)}_{1221}(-\omega,\omega,\Omega,-\Omega) + \chi^{(3)}_{1212}(-\omega,\omega,\Omega,-\Omega)\right.$$

$$\left. + X^{(3)}_{1122}(-w,w,\Omega,-\Omega)\right\}|E(\Omega)|^2\ . \qquad \text{(A.19)}$$

$$\delta n_{\perp} = \frac{12\pi}{n}\left\{\chi^{(3)}_{1122}(\omega,\omega,\Omega,-\Omega)\right\}|E(\Omega)|^2. \qquad \text{(A.20)}$$

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2. P.D. Maker and R.W. Terhune, "Study of optical effects due to induced polarization third order in the electric field strength," Phys. Rev., vol. 137, p. A801, 1965.

## APPENDIX B

## ON THE EVALUATION OF CERTAIN ELLIPTIC INTEGRALS

### A. Nonlinear DFB Structure

In the theory of bistability in periodic structures the following integrals arise:

i) Input-Output relation

$$\int_I^J \frac{dy}{[-V(y)]^{1/2}} = 2 \tag{B.1}$$

ii) Distribution of forward intensity

$$\int_I^{y(\zeta)} \frac{dy}{[-V(y)]^{1/2}} = 2\zeta \ , \tag{B.2}$$

where I and J are the normalized incident and transmitted intensities, y is the forward-going flux within the structure, $\zeta$ is the normalized distance z/L, and the function V(y) is given by

$$V(y) = (J-y)\{(\kappa L)^2 y - (y-J)(\Delta\beta L+y)^2\}. \tag{B.3}$$

Here $\kappa L$ and $\Delta\beta L$ are the coupling constant and detuning parameter respectively.

Equations (B.1) and (B.2) are elliptic integrals of the first kind. Explicit formulas for evaluating these integrals can be found in ref. 1 for example. In order to apply those formulas it is convenient to rewrite Eq. (B.1) so that one of the limits of the integral is a root of the quartic polynomial V(y). Since J is a root, and y must lie between I and J ($I > y \geq J$), we can write

$$\int_I^y = \int_I^J - \int_y^J$$

$$= 2 - \int_y^J = 2\zeta$$

Hence

$$\int_{y(\zeta)}^{J} \frac{dy}{[-V(y)]^{1/2}} = 2(1-\zeta). \qquad \text{(B.4)}$$

If all the roots ($y_1 > y_2 > y_3 > y_4$) of the quartic are real then formula 256.00 in ref. 1 gives

$$\int_{y_2}^{y} \frac{dy}{[y_1-y)(y-y_2)(y-y_3)(y-y_4)]^{1/2}} = gsn^{-1}(\sin\psi,\kappa) \quad (B.5)$$

where

$$k^2 = \frac{(y_1-y_2)(y_3-y_4)}{(y_1-y_3)(y_2-y_4)} \quad (B.6)$$

$$g = \frac{2}{[(y_1-y_3)(y_2-y_4)]^{1/2}} \quad (B.7)$$

and

$$\psi = \arcsin\left|\frac{(y_1-y_3)(y-y_2)}{(y_1-y_2)(Y-y_3)}\right|^{1/2} . \quad (B.8)$$

Here $sn^{-1}$ is the elliptic integral $F(\psi/k)$, the inverse of the elliptic function sn.

Equation (B.2) now becomes

$$gsn^{-1}(\sin\psi,k) = -2(1-\zeta),$$

which upon inversion yields

$$sn\left(\frac{-2(1-\zeta)}{g}, k\right) = \sin\psi. \quad (B.9)$$

Using (B.8), Eq. (B.9) can be written

$$sn^2\left(-2\frac{(1-\zeta)}{g}, k\right) = \frac{(y_1-y_3)(y-y_2)}{(y_1-y_2)(y-y_3)} \quad (B.10)$$

Solving for y, we finally obtain

$$y = \frac{y_3\left(\frac{y_1-y_2}{y_1-y_3}\right)\mathrm{sn}^2\left(\frac{-2(1-\zeta)}{g},\ k\right) - y_2}{\left(\frac{y_1-y_2}{y_1-y_3}\right)\mathrm{sn}^2\left(\frac{-2(1-\zeta)}{g},\ k\right) - 1} \ . \tag{B.11}$$

This is an explicit expression for the intensity of the forward-going wave within the DFB structure. It depends on the transmitted intensity through the roots $y_i(J)$ of the quartic $V(y)$.

Since $y(\zeta=0) = I$, the incident intensity is given by

$$I = \frac{y_3\left(\frac{y_1-y_2}{y_1-y_3}\right)\mathrm{sn}^2\left(\frac{-2}{g},\ k\right) - y_2}{\left(\frac{y_1-y_2}{y_1-y_3}\right)\mathrm{sn}^2\left(\frac{-2}{g},\ k\right) - 1} \ . \tag{B.12}$$

Note that I is a single valued function of the transmitted intensity J, whereas the converse is not true.

A numerical code written in HPL was developed for the evaluation of Eqs. (B.11) and (B.12). Briefly, the algorithm is as follows:

i) Pick a value of J.

ii) For this value of J, find the roots of the quartic.

iii) Use these roots to compute the quantities k and g.

iv) Use k and g in the method of Arithmetic-Geometric Mean [ 2 ] to compute the elliptic function.

v) Find I from Eq. (B.12).

vi) Go to the next J.

In evaluating Eq. (B.11), J was kept fixed and $\zeta$ varied from 0 to 1.

## B. Degenerate Four-Wave Mixing

Here the integrals of interest are

$$\int_P^J \frac{du}{[Q(u;\ J,\ P,\ B,\ \gamma,\ \Delta k/4\beta F)]^{1/2}} = 4\beta FL = \frac{\pi L}{L_c} \tag{B.13}$$

for finding the transmitted probe intensity and

$$\int_{u(\zeta)}^J \frac{du}{[Q(u)]^{1/2}} = \frac{\pi L}{L_c}(1-\zeta) \tag{B.14}$$

for finding the spatial distribution of the probe intensity. The quartic polynomial Q(u) is given by

$$Q(u) = \left(\frac{J-u}{\gamma}\right)[\gamma(1+P-u)(u-J+B)u - (J-u)(\sqrt{\gamma}\,\frac{\Delta k}{4\beta F} + 2u + B - 1 - P)^2. \tag{B.15}$$

Equation 255.00 of ref. 1 is useful for transforming the integral (B.13) into a standard Jacobi elliptic integral. Using the roots $u_i$ of the quartic Q(u), we can write

$$\int_P^J \frac{du}{[(u_1-u)(u-u_2)(u-u_3)(u-u_4)]^{1/2}} = \frac{\pi L}{L_c \sigma^{1/2}} = gF(\phi/k) \tag{B.16}$$

where $\sigma = \gamma/(\gamma-4)$, and g, $\phi$, k are defined in Eqs. (4.29)-(4.31). For a given value of probe intensity P, the transmitted signal ranges from P to $J_m$, and the integral $F(\phi/k)$ is calculated using the method of Arithmetic-Geometric Mean. When J reaches $J_m$ we obtain the complete elliptic integral $F(\phi=\pi/2/k) = K(k)$ and hence the interaction length L at this point is

$$\frac{L_m}{L_c} = \frac{\sigma^{1/2}}{\pi} K(k). \tag{B.17}$$

To continue the solution beyond $L_m$ requires that we add (or subtract) integer multiples of 2K to (or from) the fundamental solution $F(\phi/k)$. This is due to the periodic nature of the elliptic functions of which the elliptic integrals are inverses. Thus the several branches of the solution are given by $F(\phi/k)$, $2K - F(\phi/k)$, $2K + F(\phi/k)$, $4K - F(\phi/k)$ etc.

## REFERENCES -- APPENDIX B


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